Errors Cont.

Monday, April 24, 2023 8:52 AM

problem 1: $f(x) = \cos(x)$, we want estimate of $\cos(0.2)$ with error bounded by 10^{-5} (5 decimal places)

something close to actual value, but can calc

1) taylor expand $\cos(x)$ @ $x=0$ $(x=a)$ (bc we know $\cos(0)=1$) then...

$$\cos(x) \overset{x=0}{=} 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

2) find radius of convergence $\xrightarrow[\text{test}]{\text{ratio}}$ $R=\infty$ ✓ $x=0.2$ good since in $R=\infty$

$$\lim_{n\to\infty} \left| \frac{x^{2n+2}}{(2(n+1))!} \cdot \frac{(2n)!}{x^{2n}} \right| = \frac{x^2}{(n+1)} = 0 \quad R=\infty$$

3) estimate error (use taylor error thm)

---

taylor error thm: let $f(x)$ = differentiable function & $T_n(x)$ = nth taylor polynomial continued @ a

$$f(x) = \underbrace{f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n}_{T_n(x)} + \underbrace{R_n(x)}$$

error : value missing to reach exact value

then $|R_n(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \right|$ with c between x & a

c unknown : value that makes derivative reach exact remainder value

in practice : $|R_n(x)| \le \frac{|M|}{(n+1)!}(x-a)^{n+1}$ with M = max of $f^{(n+1)}(c)$

---

order 1: $\cos(x) \approx 1 - 0x$ then error $|R_1(x)| = \left| \frac{\cos^{(2)}(c)}{2!}(x-0)^2 \right| = \left| \frac{-\cos(c)}{2!}(0.2)^2 \right| \le \frac{(0.2)^2}{2!} = 0.02$ (not error of 10^{-5})

$\cos'(x) = -\sin(x)$ $\cos''(x) = -\sin'(x) = -\cos(x)$ $c \in (0, 0.2)$ (a, x)

$|\cos^{(2)}(c)| = |-\cos(c)| \le 1$ in $(0, 0.2)$

$-\cos''(x) = \sin''(x) = \cos'(x) = -\sin(x)$

order 4: $\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$, then error $|R_4(x)| = \left| \frac{\cos^{(5)}(c)}{5!}(0.2-0)^5 \right| \le \frac{1}{5!}(0.2)^5 = \frac{1}{5!}(2 \times 10^{-1})^5 = \frac{2^5 \cdot 10^{-5}}{120} = \frac{4}{15} \cdot 10^{-5}$ ✓ ← use order 4

$|\cos^{(5)}(c)| = |-\sin(c)| \le 1$

check: $\cos(0.2) = 0.98006657784\dots$ ✓

taylor $f_n = 4$: $x = 0.2 = 0.980066$ ✓

problem 2: $f(x) = \ln(1+x)$, approximate $\ln(2)$ $x=0 \to \ln(1) = 0$

1) taylor series $\ln(1+x)$ @ $x=0$ is ...

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

$\frac{d}{dx} \ln(1+x) = \frac{1}{1+x}$ (take derivative, solve taylor's, take integral)

geometric series $(r = -x) \to \sum -x^n$

taylor $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 \dots$

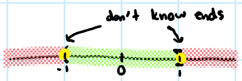
taylor $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \dots$

2) radius of convergence is $R=1$ + check endpoints

$x=-1 \to$ diverges , $x=1 \to$ converges

$\left(\sum \frac{(-1)^{n+1}(-1)^n}{n} = -\sum \frac{1}{n} : \text{p series}\right)$ one of them odd always $\left(\sum (-1)^n \frac{1}{n} : \text{alternating series test}\right)$

$$\lim_{n\to\infty} \sqrt[n]{\frac{x^n}{n}} = \frac{x}{n^{1/n}} = x < 1 \to \text{converge} \quad R=1$$



3) error estimates : $f(x) = \ln(1+x)$

order 2: $x - \frac{x^2}{2} \overset{x=1}{=} 1 - \frac{1}{2} = \frac{1}{2} = 0.5$ $\ln(1+x) \overset{x=1}{=} \ln(2)$

order 3: $x - \frac{x^2}{2} + \frac{x^3}{3} = 0.5 + \frac{1}{3} = 0.8\overline{3}$

order 4: $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} = 0.83 - \frac{1}{4} = 0.58\overline{3}$

order 5: $0.78\overline{3}$

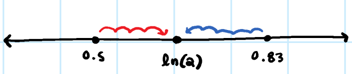
order 6: 0.616

order 7: 0.75952

order 8: 0.6345

order 9: 0.74563

order 10: 0.645634



$f^{(n+1)}(x)$ (value within error of taylor series)

$n=0$ $f'(x) = (1+x)^{-1} \to (-1)^0 \cdot 0! \, (1+x)^{-1}$

$n=1$ $f''(x) = (-1)(1+x)^{-2} \to (-1)^1 \cdot 1! \, (1+x)^{-2}$

$n=2$ $f'''(x) = (-1)(-2)(1+x)^{-3} \to (-1)^2 \cdot 2! \, (1+x)^{-3}$

$n=3$ $f^{(4)}(x) = (-1)(-2)(-3)(1+x)^{-4} \to (-1)^3 \cdot 3! \, (1+x)^{-4}$

$\vdots$

$n=n$ $f^{(n+1)}(x) = (-1)^n \cdot n! \, (1+x)^{-(n+1)} = \frac{(-1)^n n!}{(1+x)^{n+1}}$

($\ln(1+x)$) doesn't have max, so that's why we have to calc value for next derivative of $\ln(1+x)$

order 9: 0.74563

order 10: 0.645634

0.6931471806...

$n = n$ $\quad f^{(n+1)}(x) = (-1)^n \cdot n!\,(1+x)^{-(n+1)} = \frac{(-1)^n n!}{(1+x)^{n+1}}$

general error is $|R_n(x)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}\right| \longrightarrow x=1,\ a=0,\ c \in (0,1)$

given

$= \left|\frac{(-1)^n n!}{(1+c)^{n+1}} \cdot \frac{1}{(n+1)!} \cdot (1-0)^{n+1}\right| = \left|\frac{1}{(1+c)^{n+1}} \cdot \frac{1}{n+1}\right| \le \frac{1}{n+1}$

absolute value of $(-1)^n$

want error that goes to 0 & only in n's

want to be max value remainder can be → smallest denom → $c=0$

thus, to get 5 decimal digits, we need $n = 10^5 = 100000$